Journal of Approximation Theory **94**, 494–508 (1998)
Article No. AT983290–3299

BOOK REVIEWS

Book Review Editor: Walter Van Assche

Books

E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften **316**, Springer-Verlag, Berlin, 1997, xv + 505 pp.

Mathematicians (and physicists!) generally "know" Dirichlet's principle. They are likely less familiar with the related W. Thomson (Lord Kelvin) principle (in electrostatics) and its special case called the Gauss variation problem, which is the problem of minimizing the energy associated with any sourceless vector field in the region bounded by a given closed set (the conductor) over which an (electric) charge is to be distributed. As a matter of fact, there are significant differences between the two principles: Dirichlet had to prove the existence of a minimizing point function (actually a harmonic function) satisfying essential boundary conditions, whereas Gauss had only to prove the existence of a minimizing set function (actually a positive Borel measure of prescribed total mass). Powerful compactness arguments can now be used, measures and their potentials being naturally the precision tools of modern potential theory.

This book presents a self-contained and comprehensive treatment of the Gauss variation problem in the complex plane in the presence of a weight (or external field). The essential difference between earlier works (by Gauss himself and, from the 1930s, by Frostman and the Polish school headed by Leja) and the newer treatments (initiated in the 1980s by Rakhmanov and, independently, Mhaskar and Saff) lies not only in its greater generality (a weight is admissible under rather weak conditions), but also in its emphasis on determining (from the given external field) the support set of the minimum energy measure and its properties. A most glaring difference with the classical unweighted equilibrium problem is that this support need not coincide with the outer boundary of the set and, in fact, can be an arbitrary subset, possibly with positive area. This modern interaction of approximation-theoretical techniques and logarithmic potential theory with external fields proved extremely useful in recent years, so that several difficult problems could be solved (Freud's problems on the asymptotics of orthogonal polynomials and the so-called "1/9" conjecture, among others). Most striking is the fact that the external field problem (and its extension

0021-9045/98 $25.00

to signed measures) provides a unified approach to many (seemingly different) problems in constructive analysis and to several important concepts in potential theory itself.

From indispensable preliminaries (Chapter 0) on harmonic and subharmonic functions in the plane, Chapters I and II discuss in depth the Gauss variation problem (which is the basis for all later developments and concrete applications), while providing motivations and detailed proofs for many of the basic results from potential theory (such as the famous "principles"), which are presented here as needed (a listing of them, with locations in the text, is given in a useful appendix). The emphasis is naturally on the effect (similarities but very important differences!) that the external field has on the equilibrium distribution. The relationship between a Borel measure and its potential is examined in detail: recovery techniques of measures (including the Sokhotskyi–Plemelj formula for arcs and its integrated version known as the Stieltjes–Perron inversion formula of Cauchy transforms), balayage problems, Green functions, and Green potentials.

Chapter III exploits, in the weighted setting throughout, the intimate connection of logarithmic potentials with complex polynomials (whose logarithm of the modulus is indeed the negative of the potential of the counting measure on their zeros). Since the Fekete points (known explicitly in certain simple cases) minimize the energy for discrete measures, it is not surprising that their asymptotic distribution is nothing but the required equilibrium distribution. The classical identity between logarithmic capacity, transfinite diameter and Chebyshev constant must be modified slightly (especially the latter) in the weighted case. Other modifications are that the zeros of the Chebyshev polynomials (or, more generally, of monic polynomials with asymptotically minimal norm) accumulate on the polynomial convex hull of the support of the equilibrium measure, which is also the smallest compact set on which the (essential) supremum norm actually lives. In many extremal problems, Chebyshev polynomials can serve as a prototype for comparison (the main concern here is the asymptotic behavior of those polynomials and their zeros).

Chapter IV discusses methods for determining the extremal measure. Finding its support is one of the most important aspects of the energy problem (it then remains to solve Dirichlet problems and to launch the recovery machinery). As discovered by Mhaskar–Saff, this amounts to minimizing, over the set of possible supports, the (quasi-everywhere) constant value of the extremal potential. It is surprising that such an obviously hard problem can be solved explicitly under suitable convexity assumptions (satisfied by the important weights of Freud, Jacobi, Laguerre, and also by certain radially symmetric weights), the required support then being an interval whose endpoints can be obtained by solving a (simple) integral equation.

In Chapter V, the fact that the weighted equilibrium problem solves a certain Dirichlet problem, coupled with the fact that the weighted Fekete points (and the so-called Leja points, which are easier to determine in view of their adaptive nature) are distributed according to the equilibrium distribution, leads to a numerically stable method for determining Dirichlet solutions (or Green functions and conformal maps of simply connected domains).

Chapter VI deals with weights on the real line and discusses the possibility of (weighted) approximation, fast decreasing polynomials, the discretization of logarithmic potentials, norm inequalities, and n-widths. Some problems receive interesting explicit solutions here.

A few special subjects of the asymptotic theory of orthogonal polynomials (zero distribution, strong asymptotics for the leading coefficients) are examined in Chapter VII, which is mainly a survey on Freud weights with tribute to Ullman's theory.

Finally Chapter VIII, Signed Measures, is about condensers, i.e., arrangements of supports of measures of different signs, important for *rational approximation*. It contains basic theorems for equilibrium potentials and measures, rational Fekete points, a treatment of a Zolotarjov (sometimes known to Western readers as Zolotarev) problem, conformal mapping of ring domains, and a discrepancy theorem for zeros of polynomials.

There is an appendix on the Dirichlet problem and harmonic measures, another appendix (by Thomas Bloom) on weighted approximation in $\mathbb{C}^N$, an index of basic results of potential theory, a bibliography of 240 items, a list of symbols, and a regular index.

In conclusion, this monograph on potential theory in modern approximation theory (with notes and historical references) is most remarkable. It is indeed up-to-date, essentially comprehensive, and extremely well written. The fact that all the proofs are explicitly given, separately from the always clearly formulated statements, is evidently welcome: the hurried reader may decide to skip many of the proofs and concentrate rather on the results (many of them appear here for the first time), while the interested student in mathematics can exercise at will. However, for the (non-empty!) set of possible readers who still consider that physical mathematics in the sense of Sommerfeld (i.e., the physical motivation of mathematical methods) remains fruitful, the quasi-total lack of such intuitive connections may prove somewhat frustrating. In our opinion, the practical usefulness of the book would have been enhanced if (mostly nontrivial) electrostatic interpretations had been provided more often. For example, a sentence such as the following: "for unbounded regions... during the sweeping out process, the potential increases by a constant" (p. 110) is worth some complementary substantiation of this kind. Similarly, Example 4.8 (p. 118) can be interpreted concretely by saying that it yields the charge induced by an external potential on an insulated (charged) conductor. Also, a Green potential is simply the potential of the total field created by internal charges and counter charges induced on the grounded conductors bounding the domain. It is quite surprising that the underlying, though ubiquitous, concept of (electric) induction is not even mentioned; the more so as the authors consider that "the physical interpretation (of the contact problem of elasticity, see p. 247) readily explains many of the theorems in this monograph, and is a useful device for making reasonable guesses on the behavior of weighted potentials."

Jean Meinguet
E-mail: Meinguet@anma.ucl.ac.be
ARTICLE NO. AT983290

H. N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, Series in Approximations and Decompositions **7**, World Scientific, Singapore, 1996, xiv + 379 pp.

The theory of approximation of functions by trigonometric and algebraic polynomials provides a role model for the study of other approximation processes, such as splines, rational functions, and wavelets. The main theme of this excellently written monograph is the approximation of functions on the whole real line by algebraic polynomials. At least for 40 years, many people worked on the classical Bernstein approximation problem which seeks conditions on the weight w such that the set of functions $\{w(x)\,x^k\}_{k=0}^{\infty}$ is fundamental in the class of continuous functions on the real line, vanishing at infinity. In the 1970s Géza Freud started to develop a rich theory for weighted polynomial approximation, first for the weight $w(x) = \exp(-x^2/2)$, later for general weights. Since then, many people have contributed deep results using a variety of different techniques and ideas. There are other excellent books which describe adequately these achievements, mainly from the perspective of the potential theory or orthogonal polynomials. But the main thrust of this book is to introduce the subject from an approximation theory point of view. Therefore the author first treats the basic topics of approximation on a bounded interval, such as interpolation and quadrature in Chapter 1, Favard-type estimates, K-functional, degree of approximation in Chapter 2. Later, one of the main motivations is to study analogs for the weighted polynomial approximation on the real line.

Chapter 3 develops many technical estimates regarding the "Freud polynomials." In Chapter 4, the direct and converse theorems for the degree of weighted approximation on the real axis,

$$E_{p,n}(w; f) := \inf\{\, \|w(f-P)\|_p,\ P \in \Pi_n\}$$